数理解析研究所講究録
1189 巻 2001 年 105-121

Powers of class $w\mathrm{A}(s,t)$ operators associated with generalized Aluthge transformation

東京理科大 理　柳田 昌宏 (Masahiro Yanagida)
Faculty of Science, Science University of Tokyo

Abstract

This report is based on the following preprint:

M.Yanagida, *Powers of class wA(s, t) operators associated with generalized Aluthge transformation*, to appear in J. Inequal. Math.

An operator $T=U|T|$ is said to belong to class $w\mathrm{A}(s,t)$ for $s,t>0$ if $|\tilde{T}_{s,t}|^{\frac{2t}{s+t}}\geq|T|^{2t}$ and $|T|^{2s}\geq|(\tilde{T}_{s,t})^{*}|^{\frac{2s}{s+t}}$, where $\tilde{T}_{s,t}=|T|^{s}U|T|^{t}$. We show that if T belongs to class $w\mathrm{A}(s,t)$, then T^{n} belongs to class $w\mathrm{A}(\frac{s}{n},\frac{t}{n})$ for every natural number n.

1 Introduction

1.1 An order preserving operator inequality

In this report, an operator means a bounded linear operator on a Hilbert space H. An operator T is said to be positive (denoted by $T\geq 0$) if $(Tx,x)\geq 0$ for all $x\in H$, and also T is said to be strictly positive (denoted by $T>0$) if T is positive and invetible.

We begin this report by introducing the following result which is quite useful for the study of the class of operators including normal operators ($\Longleftrightarrow T^{*}T=TT^{*}$).

Theorem F (Furuta inequality [12]).

If $A\geq B\geq 0$, then for each $r\geq 0$,

(i) $$(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{1}{q}}\geq(B^{\frac{r}{2}}B^{p}B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^{p}A^{\frac{r}{2}})^{\frac{1}{q}}\geq(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p\geq 0$ and $q\geq 1$ with $(1+r)q\geq p+r$.

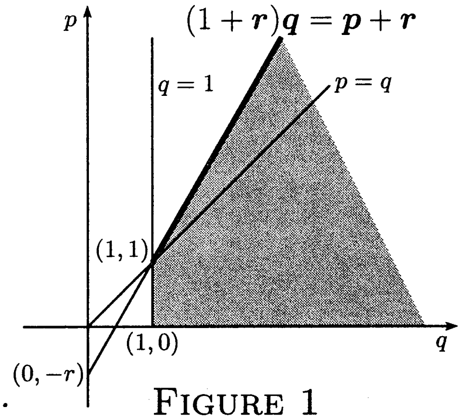

FIGURE 1

We remark that Theorem F yields Löwner-Heinz theorem "$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$" when we put $r = 0$ in (i) or (ii) stated above. Alternative proofs of Theorem F are given in [10][23] and also an elementary one-page proof in [13]. It is shown in [25] that the domain drawn for p, q and r in Figure 1 is the best possible for Theorem F.

1.2 Aluthge transformation of p-hyponormal and log-hyponormal perators

An operator T is said to be p-hyponormal for $p > 0$ if $(T^*T)^p \geq (TT^*)^p$, and T is said to be log-hyponormal if T is invertible and $\log T^*T \geq \log TT^*$. p-Hyponormality and log-hyponormality were defined as extensions of hyponormality, that is, $T^*T \geq TT^*$. It is easily seen that every q-hyponormal operator is p-hyponormal for $q \geq p > 0$ by Löwner-Heinz theorem, and every invertible p-hyponormal operator for some $p > 0$ is log-hyponormal since $\log t$ is an operator monotone function. We remark that p-hyponormality tends to log-hyponormality as $p \to +0$ since $\frac{X^p - I}{p} \to \log X$ as $p \to +0$ for every positive operator X.

The operator $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is called Aluthge transformation of an operator T whose polar decomposition is $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$. Aluthge transformation was first introduced by Aluthge [1], and he showed the following result on Aluthge transformation of p-hyponormal operators as an application of Theorem F.

Theorem A **([1]).** *Let $T = U|T|$ be the polar decomposition of a p-hyponormal operator for $0 < p < 1$ and U be unitary. Then*

(i) $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ *is $(p+\frac{1}{2})$-hyponormal if* $0 < p \leq \frac{1}{2}$.

(ii) $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ *is hyponormal if* $\frac{1}{2} \leq p < 1$.

We remark that $\sigma(\tilde{T}) = \sigma(T)$ holds for any operator T [4][7], and Theorem A states that $\tilde{T}$ belongs to a smaller class than a p-hyponormal operator T for $0 < p < 1$.

A generalization of Aluthge transformation of an operator $T = U|T|$ is $\tilde{T}_{s,t} = |T|^s U|T|^t$ for $s > 0$ and $t > 0$. In fact, it is clear that $\tilde{T}_{\frac{1}{2},\frac{1}{2}} = \tilde{T}$. Huruya [19] and Yoshino [29] showed an extension of Theorem A on generalized Aluthge transformation of p-hyponormal operators. Tanahashi [26] showed a parallel result on generalized Aluthge transformation of log-hyponormal operators.

1.3 Classes of operators associated with Aluthge transformation

Recently, Aluthge and Wang introduced the class of w-hyponormal operators via Aluthge transformation $\tilde{T}$ in [4], and showed an equivalent condition to w-hyponormality in [5].

Definition ([4][5]).

$$T : w\text{-hyponormal} \Longleftrightarrow |\tilde{T}| \geq |T| \geq |(\tilde{T})^*|$$
$$\Longleftrightarrow (|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |T^*| \text{ and } |T| \geq (|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}})^{\frac{1}{2}},$$

where $\tilde{T}$ is Aluthge transformation of T.

As a generalization of the class of w-hyponormal operators, Ito [20] introduced class $w\mathrm{A}(s,t)$ for $s > 0$ and $t > 0$ via generalized Aluthge transformation $\tilde{T}_{s,t}$. In fact, it is clear that class $w\mathrm{A}(\frac{1}{2},\frac{1}{2})$ coincides with the class of w-hyponormal operators.

Definition ([20]). For $s > 0$ and $t > 0$,

$$T \in \text{class } w\mathrm{A}(s,t) \Longleftrightarrow |\tilde{T}_{s,t}|^{\frac{2t}{s+t}} \geq |T|^{2t} \text{ and } |T|^{2s} \geq |(\tilde{T}_{s,t})^*|^{\frac{2s}{s+t}}$$
$$\Longleftrightarrow (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t} \text{ and } |T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}},$$

where $\tilde{T}_{s,t}$ is generalized Aluthge transformation of T. For the sake of convenience, we call class $w\mathrm{A}(1,1)$ class $w\mathrm{A}$ for short.

He also pointed out the following fact.

Proposition B ([20]). *$T \in$ class $w\mathrm{A} \Longleftrightarrow |T^2| \geq |T|^2$ and $|T^*|^2 \geq |T^{2*}|$.*

1.4 Related classes and their inclusion relations

On the other hand, Furuta, Ito and Yamazaki [15] introduced a class of operators called class A.

Definition ([15]). $T \in$ class A $\Longleftrightarrow |T^2| \geq |T|^2$.

They showed that every log-hyponormal operator belongs to class A and every class A operator is paranormal ($\Longleftrightarrow \|T^2x\| \geq \|Tx\|^2$ for every unit vector x). This relations give another proof of the result by Ando [6].

As a generalization of class A, Fujii, D.Jung, S.H.Lee, M.Y.Lee and Nakamoto [11] introduced class A(s,t) for $s>0$ and $t>0$. In fact, it was pointed out in [28] that class A$(1,1)$ coincides with class A.

Definition ([11]). For $s>0$ and $t>0$,

(i) $T \in$ class A$(s,t) \Longleftrightarrow (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$.

(ii) $T \in$ class AI$(s,t) \Longleftrightarrow T \in$ class A(s,t) and T is invertible.

We remark the following inclusion relations:

$$(\spadesuit) \qquad \text{class A}(s,t) \supseteq \text{class } w\text{A}(s,t) \supseteq \text{class AI}(s,t)$$

holds for each $s>0$ and $t>0$. The first relation of $(\spadesuit)$ holds obviously, and the second holds by the following lemma.

Lemma F ([14]). *Let $A>0$ and B be an invertible operator. Then*

$$(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

holds for any real number λ.

In fact, the first inequality in the definition of class wA(s,t) yields the second by applying Lemma F in case T is invertible as follows:

$$\begin{aligned}
&(|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}} \\
&= |T|^s|T^*|^t(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{-t}{s+t}}|T^*|^t|T|^s && \text{by Lemma F} \\
&\leq |T|^s|T^*|^t \qquad |T^*|^{-2t} \qquad |T^*|^t|T|^s && \text{by the first inequality} \\
&= |T|^{2s}.
\end{aligned}$$

We also remark the following results.

Theorem C.1 **([20]).**

(i) *If an operator T is p-hyponormal for some $p > 0$ or log-hyponormal, then T belongs to class $wA(s, t)$ for all $s > 0$ and $t > 0$.*

(ii) *Every class $wA(s_1, t_1)$ operator belongs to class $wA(s_2, t_2)$ for each $0 < s_1 \le s_2$ and $0 < t_1 \le t_2$.*

Theorem C.2 **([11]).**

(i) *An operator T is log-hyponormal if and only if T belongs to class $AI(s, t)$ for all $s > 0$ and $t > 0$.*

(ii) *Every class $A(s, t_1)$ operator belongs to class $A(s, t_2)$ for each $0 < t_1 \le t_2$.*

The following diagram shows the inclusion relations among the classes of operators mentioned above.

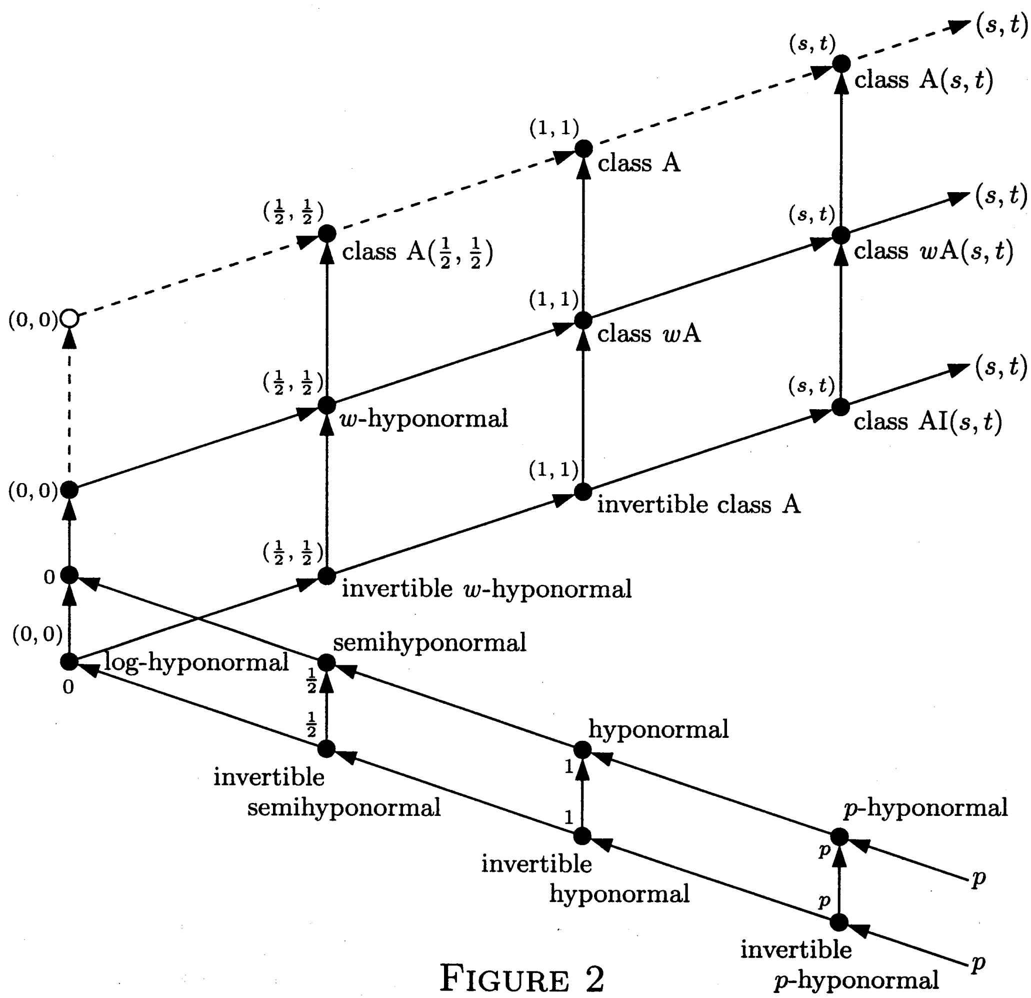

FIGURE 2

1.5 Results on powers of non-normal operators

Recently, Aluthge and Wang showed results on powers of p-hyponormal and log-hyponormal operators in [2][3]. Extensions of the results were shown by Furuta and Yanagida [16][17], Ito [22] and Yamazaki [27].

As continuation of this study, Aluthge and Wang [5] showed the following result on powers of invertible w-hyponormal operators. A simplified proof of Theorem D.1 was given by Y.O.Kim [24].

Theorem D.1 **([5]).** *Let T be an invertible w-hyponormal operator. Then T^2 is also w-hyponormal.*

Cho, Huruya and Y.O.Kim [8] showed the following result which states that Theorem D.1 remains valid with a weaker condition $N(T) = \{0\}$ than the invertibility of T.

Theorem D.2 **([8]).** *Let T be a w-hyponormal operator with $N(T) = \{0\}$. Then T^2 is also w-hyponormal.*

On the other hand, Ito [21] showed the following result on powers of invertible class A operators.

Theorem D.3 **([21]).** *Let T be an invertible class A operator. Then the following assertions hold for all positive integer n:*

(i) $|T^{n+1}|^{\frac{2n}{n+1}} \geq |T^n|^2$ *and* $|T^{n*}|^2 \geq |T^{n+1*}|^{\frac{2n}{n+1}}$.

(ii) $|T^n|^{\frac{2}{n}} \geq \cdots \geq |T^2| \geq |T|^2$ *and* $|T^*|^2 \geq |T^{2*}| \geq \cdots \geq |T^{n*}|^{\frac{2}{n}}$.

(iii) $|T^{2n}| \geq |T^n|^2$ *and* $|T^{n*}|^2 \geq |T^{2n*}|$, *i.e., T^n also belongs to class A.*

As an extension of both Theorem D.1 and (iii) of Theorem D.3, Yamazaki [28] showed the following result on powers of class AI(s,t) operators.

Theorem D.4 **([28]).** *Let T be a class $AI(s,t)$ operator for $s \in (0,1]$ and $t \in (0,1]$. Then T^n belongs to $AI(\frac{s}{n}, \frac{t}{n})$ for all positive integer n.*

In fact, Theorem D.4 yields Theorem D.1 by putting $s = t = \frac{1}{2}$ and $n = 2$ since class AI$(\frac{1}{4}, \frac{1}{4}) \subseteq$ class AI$(\frac{1}{2}, \frac{1}{2})$ by (ii) of Theorem C.1. Theorem D.4 also yields (iii) of Theorem D.3 by putting $s = t = 1$ since class AI$(\frac{1}{n}, \frac{1}{n}) \subseteq$ class AI$(1, 1)$ by (ii) of Theorem C.1. It is interesting to remark that Theorem D.4 states that T^n belongs to a smaller class than a class AI(s, t) operator T for $s \in (0, 1]$ and $t \in (0, 1]$.

In this report, we shall show several results on powers of class wA(s, t) operators as extensions of the results on powers of class AI(s, t) operators and w-hyponormal operators mentioned above.

2 Results

Firstly, we show the following result on powers of class wA operators.

Theorem 1. *Let T be a class wA operator. Then the following assertions hold for all positive integer n:*

(i) $|T^{n+1}|^{\frac{2n}{n+1}} \geq |T^n|^2$ *and* $|T^{n*}|^2 \geq |T^{n+1*}|^{\frac{2n}{n+1}}$.

(ii) $|T^n|^{\frac{2}{n}} \geq \cdots \geq |T^2| \geq |T|^2$ *and* $|T^*|^2 \geq |T^{2*}| \geq \cdots \geq |T^{n*}|^{\frac{2}{n}}$.

Secondly, we show the following result on powers of class wA(s, t) operators.

Theorem 2. *Let T be a class $wA(s, t)$ operator for $s \in (0, 1]$ and $t \in (0, 1]$. Then T^n belongs to $wA(\frac{s}{n}, \frac{t}{n})$ for all positive integer n.*

Theorem 1 and Theorem 2 are extensions of Theorem D.3 and Theorem D.4, respectively, since every class AI(s, t) operator belongs to class wA(s, t) by (♠). In other words, Theorem 1 and Theorem 2 state that Theorem D.3 and Theorem D.4 remain valid for class wA and class wA(s, t) operators without the invertibility of T, respectively.

Theorem 2 yields the following result as an immediate corollary which is an extension of Theorem D.2.

Corollary 3. *Let T be a w-hyponormal operator. Then T^n is also w-hyponormal for all positive integer n.*

3 Proofs of the results

In order to give a proof of Theorem 1, we prepare the following results.

Proposition 4. *Let A and B be positive operators. Then the following assertions hold:*

(i) *If $(B^{\frac{\beta_0}{2}}A^{\alpha_0}B^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}} \geq B^{\beta_0}$ holds for fixed $\alpha_0 > 0$ and $\beta_0 > 0$, then*

$$(3.1) \qquad (B^{\frac{\beta}{2}}A^{\alpha_0}B^{\frac{\beta}{2}})^{\frac{\beta}{\alpha_0+\beta}} \geq B^{\beta}$$

holds for any $\beta \geq \beta_0$, and

$$(3.2) \qquad A^{\frac{\alpha_0}{2}}B^{\beta_1}A^{\frac{\alpha_0}{2}} \geq (A^{\frac{\alpha_0}{2}}B^{\beta_2}A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0+\beta_1}{\alpha_0+\beta_2}}$$

holds for any β_1 and β_2 such that $\beta_2 \geq \beta_1 \geq \beta_0$.

(ii) *If $A^{\alpha_0} \geq (A^{\frac{\alpha_0}{2}}B^{\beta_0}A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta_0}}$ holds for fixed $\alpha_0 > 0$ and $\beta_0 > 0$, then*

$$A^{\alpha} \geq (A^{\frac{\alpha}{2}}B^{\beta_0}A^{\frac{\alpha}{2}})^{\frac{\alpha}{\alpha+\beta_0}}$$

holds for any $\alpha \geq \alpha_0$, and

$$(B^{\frac{\beta_0}{2}}A^{\alpha_2}B^{\frac{\beta_0}{2}})^{\frac{\alpha_1+\beta_0}{\alpha_2+\beta_0}} \geq B^{\frac{\beta_0}{2}}A^{\alpha_1}B^{\frac{\beta_0}{2}}$$

holds for any α_1 and α_2 such that $\alpha_2 \geq \alpha_1 \geq \alpha_0$.

Lemma 5. *Let A, B and C be positive operators. Then the following assertions holds for each $p \geq 0$ and $r \in (0,1]$:*

(i) *If $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ and $B \geq C$, then $(C^{\frac{r}{2}}A^pC^{\frac{r}{2}})^{\frac{r}{p+r}} \geq C^r$.*

(ii) *If $A \geq B$, $B^r \geq (B^{\frac{r}{2}}C^pB^{\frac{r}{2}})^{\frac{r}{p+r}}$ and the condition*

$$(*) \qquad \text{if } \lim_{n\to\infty} B^{\frac{1}{2}}x_n = 0 \text{ and } \lim_{n\to\infty} A^{\frac{1}{2}}x_n \text{ exists, then } \lim_{n\to\infty} A^{\frac{1}{2}}x_n = 0$$

hold, then $A^r \geq (A^{\frac{r}{2}}C^pA^{\frac{r}{2}})^{\frac{r}{p+r}}$.

Proof of Proposition 4.

Proof of (i). Put $A_1 = (B^{\frac{\beta_0}{2}} A^{\alpha_0} B^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}}$ and $B_1 = B^{\beta_0}$, then $A_1 \geq B_1 \geq 0$ by the hypothesis. By applying (i) of Theorem F to A_1 and B_1, we have

$$(3.3) \qquad (B_1^{\frac{r_1}{2}} A_1^{p_1} B_1^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}} \geq B_1^{1+r_1} \text{ for any } p_1 \geq 1 \text{ and } r_1 \geq 0.$$

Put $p_1 = \frac{\alpha_0+\beta_0}{\beta_0} \geq 1$ and $\beta = (1+r_1)\beta_0 \geq \beta_0$ in (3.3), then we have

$$(3.1) \qquad (B^{\frac{\beta}{2}} A^{\alpha_0} B^{\frac{\beta}{2}})^{\frac{\beta}{\alpha_0+\beta}} \geq B^{\beta} \text{ for any } \beta \geq \beta_0.$$

By applying Löwner-Heinz theorem to (3.1), we have

$$(3.4) \quad (B^{\frac{\beta}{2}} A^{\alpha_0} B^{\frac{\beta}{2}})^{\frac{v}{\alpha_0+\beta}} \geq B^{v} \text{ for any } \beta \geq \beta_0 \text{ and } v \text{ such that } \beta \geq v \geq 0.$$

Put $f_{\beta_1}(\beta) = (A^{\frac{\alpha_0}{2}} B^{\beta} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0+\beta_1}{\alpha_0+\beta}}$. For any β, β_1 and v such that $\beta \geq \beta_1 \geq \beta_0$ and $\beta \geq v \geq 0$, we have

$$\begin{aligned}
f_{\beta_1}(\beta) &= (A^{\frac{\alpha_0}{2}} B^{\beta} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0+\beta_1}{\alpha_0+\beta}} \\
&= \{(A^{\frac{\alpha_0}{2}} B^{\beta} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0+\beta+v}{\alpha_0+\beta}}\}^{\frac{\alpha_0+\beta_1}{\alpha_0+\beta+v}} \\
&= \{A^{\frac{\alpha_0}{2}} B^{\frac{\beta}{2}} (B^{\frac{\beta}{2}} A^{\alpha_0} B^{\frac{\beta}{2}})^{\frac{v}{\alpha_0+\beta}} B^{\frac{\beta}{2}} A^{\frac{\alpha_0}{2}}\}^{\frac{\alpha_0+\beta_1}{\alpha_0+\beta+v}} \\
&\geq \{A^{\frac{\alpha_0}{2}} B^{\frac{\beta}{2}} \qquad B^{v} \qquad B^{\frac{\beta}{2}} A^{\frac{\alpha_0}{2}}\}^{\frac{\alpha_0+\beta_1}{\alpha_0+\beta+v}} \\
&= (A^{\frac{\alpha_0}{2}} B^{\beta+v} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0+\beta_1}{\alpha_0+\beta+v}} \\
&= f_{\beta_1}(\beta+v).
\end{aligned}$$

The above inequality holds by (3.4) and Löwner-Heinz theorem since $\frac{\alpha_0+\beta_1}{\alpha_0+\beta+v} \in [0,1]$. Therefore for each $\beta_1 \geq \beta_0$, $f_{\beta_1}(\beta)$ is decreasing for $\beta \geq \beta_1$, so that

$$A^{\frac{\alpha_0}{2}} B^{\beta_1} A^{\frac{\alpha_0}{2}} = f_{\beta_1}(\beta_1) \geq f_{\beta_1}(\beta_2) = (A^{\frac{\alpha_0}{2}} B^{\beta_2} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0+\beta_1}{\alpha_0+\beta_2}}$$

holds for any β_1 and β_2 such that $\beta_2 \geq \beta_1 \geq \beta_0$, hence we have (3.2).

(ii) can be proved in the same way as (i), so that we omit the proof. □

Lemma 5 can be obtained as an application of the following results.

Theorem E.1 **([9]).** *Let A and B be bounded linear operators on a Hilbert space H. The following statements are equivalent;*

(1) $R(A) \subseteq R(B)$*;*

(2) $AA^* \leq \lambda^2 BB^*$ *for some* $\lambda \geq 0$*; and*

(3) *there exists a bounded linear operator* C *on* H *so that* $A = BC$*.*

Moreover, if (1), (2) and (3) are valid, then there exists a unique operator C *so that*

(a) $\|C\|^2 = \inf\{\mu | AA^* \leq \mu BB^*\}$*;*

(b) $N(A) = N(C)$*; and*

(c) $R(C) \subseteq \overline{R(B^*)}$.

Theorem E.2 **([18]).** *Let X and A be bounded linear operators on a Hilbert space H. We suppose that $X \geq 0$ and $\|A\| \leq 1$. If f is an operator monotone function defined on $[0, \infty)$ such that $f(0) \leq 0$, then*

$$A^* f(X) A \leq f(A^* X A).$$

We remark that the condition (c) in Theorem E.1 is equivalent to the condition (c') $\overline{R(C)} \subseteq \overline{R(B^*)}$. Here we consider when the equality of (c') holds.

Lemma 6. *Let A and B be operators which satisfy (1), (2) and (3) of Theorem E.1, and C be the operator which is given in (3) and determined uniquely by (a), (b) and (c) of Theorem E.1. Then the following assertions are mutually equivalent:*

(i) $\overline{R(C)} = \overline{R(B^*)}$.

(ii) *If* $\lim_{n\to\infty} A^* x_n = 0$ *and* $\lim_{n\to\infty} B^* x_n$ *exists, then* $\lim_{n\to\infty} B^* x_n = 0$.

Proof. (i) is equivalent to $N(C^*) = N(B)$ and

$$N(C^*) = N(B) \oplus (N(B)^\perp \cap N(C^*)) = N(B) \oplus (\overline{R(B^*)} \cap N(C^*))$$

since $N(C^*) \supseteq N(B)$ by (c) of Theorem E.1, so that (i) is equivalent to the following (3.5):

$$\overline{R(B^*)} \cap N(C^*) = \{0\}. \tag{3.5}$$

Noting that when $y = \lim_{n\to\infty} B^* x_n$ for some $\{x_n\} \subseteq H$,

$$C^* y = C^* \left(\lim_{n\to\infty} B^* x_n\right) = \lim_{n\to\infty} C^* B^* x_n = \lim_{n\to\infty} A^* x_n$$

holds by (3) of Theorem E.1, so that we have

$$\begin{aligned}
&\overline{R(B^*)} \cap N(C^*) \\
&= \{y \mid \text{there exists } \{x_n\} \subseteq H \text{ such that } y = \lim_{n\to\infty} B^* x_n \text{ and } C^* y = 0\} \\
&= \{y \mid \text{there exists } \{x_n\} \subseteq H \text{ such that } y = \lim_{n\to\infty} B^* x_n \text{ and } \lim_{n\to\infty} A^* x_n = 0\},
\end{aligned}$$

hence (3.5) is equivalent to (ii). □

We also require the following lemma in order to give a proof of Lemma 5.

Lemma 7. *Let S be a positive operator and $\alpha \in (0,1]$. If $\lim_{n\to\infty} S x_n = 0$ and $\lim_{n\to\infty} S^\alpha x_n$ exists, then $\lim_{n\to\infty} S^\alpha x_n = 0$.*

Proof. $\lim_{n\to\infty} S^\alpha x_n \in \overline{R(S^\alpha)} \cap N(S^{1-\alpha}) = \overline{R(S)} \cap N(S) = \{0\}$ for $\alpha \in (0,1)$ since $S^{1-\alpha}\left(\lim_{n\to\infty} S^\alpha x_n\right) = \lim_{n\to\infty} S x_n = 0$ by the hypothesis. □

Proof of Lemma 5.

Proof of (i). $B \geq C$ ensures $B^r \geq C^r$ for $r \in (0,1]$ by Löwner-Heinz theorem. By Theorem E.1, there exists an operator X such that

$$B^{\frac{r}{2}} X = X^* B^{\frac{r}{2}} = C^{\frac{r}{2}}, \tag{3.6}$$

$$\|X\| \leq 1. \tag{3.7}$$

Then we have

$$\begin{aligned}
(C^{\frac{r}{2}}A^pC^{\frac{r}{2}})^{\frac{r}{p+r}} &= (X^*B^{\frac{r}{2}}A^pB^{\frac{r}{2}}X)^{\frac{r}{p+r}} && \text{by (3.6)}\\
&\geq X^*(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}}X && \text{by Theorem E.2 and (3.7)}\\
&\geq X^*B^rX && \text{by the hypothesis}\\
&= C^r && \text{by (3.6).}
\end{aligned}$$

Proof of (ii). $A \geq B$ ensures $A^r \geq B^r$ for $r \in (0,1]$ by Löwner-Heinz theorem. By Theorem E.1, there exists an operator Y such that

(3.8) $$A^{\frac{r}{2}}Y = Y^*A^{\frac{r}{2}} = B^{\frac{r}{2}},$$

(3.9) $$\|Y\| \leq 1.$$

Then we have

$$\begin{aligned}
Y^*(A^{\frac{r}{2}}C^pA^{\frac{r}{2}})^{\frac{r}{p+r}}Y &\leq (Y^*A^{\frac{r}{2}}C^pA^{\frac{r}{2}}Y)^{\frac{r}{p+r}} && \text{by Theorem E.2 and (3.9)}\\
&= (B^{\frac{r}{2}}C^pB^{\frac{r}{2}})^{\frac{r}{p+r}} && \text{by (3.8)}\\
&\leq B^r && \text{by the hypothesis}\\
&= Y^*A^rY && \text{by (3.8),}
\end{aligned}$$

so that $A^r \geq (A^{\frac{r}{2}}C^pA^{\frac{r}{2}})^{\frac{r}{p+r}}$ holds on $\overline{R(Y)}$. On the other hand, $(*)$ implies the following condition:

$(**)$ if $\lim_{n\to\infty} B^{\frac{r}{2}}x_n = 0$ and $\lim_{n\to\infty} A^{\frac{r}{2}}x_n$ exists, then $\lim_{n\to\infty} A^{\frac{r}{2}}x_n = 0$

since if $\lim_{n\to\infty} B^{\frac{r}{2}}x_n = 0$ and $\lim_{n\to\infty} A^{\frac{r}{2}}x_n$ exists, then

$$\lim_{n\to\infty} B^{\frac{1}{2}}x_n = B^{\frac{1-r}{2}}\left(\lim_{n\to\infty} B^{\frac{r}{2}}x_n\right) = 0$$

and $\lim_{n\to\infty} A^{\frac{1}{2}}x_n = A^{\frac{1-r}{2}}\left(\lim_{n\to\infty} A^{\frac{r}{2}}x_n\right)$ exists, so that $\lim_{n\to\infty} A^{\frac{1}{2}}x_n = 0$ by $(*)$, and $\lim_{n\to\infty} A^{\frac{r}{2}}x_n = 0$ by Lemma 7. $(**)$ ensures $\overline{R(Y)} = \overline{R(A^{\frac{r}{2}})}$ by Lemma 6, hence we have

$$N((A^{\frac{r}{2}}C^pA^{\frac{r}{2}})^{\frac{r}{p+r}}) = N(A^{\frac{r}{2}}C^pA^{\frac{r}{2}}) \supseteq N(A^{\frac{r}{2}}) = N(A^r) = N(Y^*),$$

so that $A^r = (A^{\frac{r}{2}}C^pA^{\frac{r}{2}})^{\frac{r}{p+r}} = 0$ on $N(Y^*)$. Consequently the proof is complete since $H = \overline{R(Y)} \oplus N(Y^*)$. $\square$

Proof of Theorem 1. Put $A_n = |T^n|^{\frac{2}{n}}$ and $B_n = |T^{n*}|^{\frac{2}{n}}$ for each integer n. By the definition, T belongs to class wA if and only if

$$(3.10) \qquad (B_1^{\frac{1}{2}} A_1 B_1^{\frac{1}{2}})^{\frac{1}{2}} = (|T^*||T|^2|T^*|)^{\frac{1}{2}} \geq |T^*|^2 = B_1$$

and

$$(3.11) \qquad A_1 = |T|^2 \geq (|T||T^*|^2|T|)^{\frac{1}{2}} = (A_1^{\frac{1}{2}} B_1 A_1^{\frac{1}{2}})^{\frac{1}{2}}.$$

We shall prove

$$(3.12) \qquad A_{n+1}^n = |T^{n+1}|^{\frac{2n}{n+1}} \geq |T^n|^2 = A_n^n$$

and

$$(3.13) \qquad B_n^n = |T^{n*}|^2 = |T^{n+1*}|^{\frac{2n}{n+1}} = B_{n+1}^n$$

hold for all positive integer n by induction. (3.12) and (3.13) hold for $n = 1$ by Proposition B. Assume (3.12) holds for $n = 1, 2, \cdots, k-1$. Then $A_{n+1} \geq A_n$ holds by Löwner-Heinz theorem for $\frac{1}{n} \in [0, 1]$, so that we have

$$(3.14) \qquad A_k \geq A_{k-1} \geq \cdots \geq A_2 \geq A_1.$$

We remark that A_1 and A_k satisfy the condition

$$(\bigstar) \qquad \text{if } \lim_{n\to\infty} A_1^{\frac{1}{2}} x_n = 0 \text{ and } \lim_{n\to\infty} A_k^{\frac{1}{2}} x_n \text{ exists, then } \lim_{n\to\infty} A_k^{\frac{1}{2}} x_n = 0$$

since

$$\begin{aligned} \lim_{n\to\infty} A_1^{\frac{1}{2}} x_n = 0 &\Longleftrightarrow \lim_{n\to\infty} |T| x_n = 0 \Longleftrightarrow \lim_{n\to\infty} T x_n = 0 \Longrightarrow \lim_{n\to\infty} T^k x_n = 0 \\ &\Longleftrightarrow \lim_{n\to\infty} |T^k| x_n = 0 \Longleftrightarrow \lim_{n\to\infty} A_k^{\frac{k}{2}} x_n = 0 \Longrightarrow \lim_{n\to\infty} A_k^{\frac{1}{2}} x_n = 0. \end{aligned}$$

The last implication holds by Lemma 7. By applying (ii) of Lemma 5 to (3.11) and (3.14), we have

$$(3.15) \qquad A_k \geq (A_k^{\frac{1}{2}} B_1 A_k^{\frac{1}{2}})^{\frac{1}{2}}.$$

By applying (ii) of Proposition 4 to (3.15),

$$(B_1^{\frac{1}{2}}A_k^{\alpha_2}B_1^{\frac{1}{2}})^{\frac{\alpha_1+1}{\alpha_2+1}} \geq B_1^{\frac{1}{2}}A_k^{\alpha_1}B_1^{\frac{1}{2}} \tag{3.16}$$

holds for any α_1 and α_2 such that $\alpha_2 \geq \alpha_1 \geq 1$, so that we have

$$(B_1^{\frac{1}{2}}A_k^kB_1^{\frac{1}{2}})^{\frac{k}{k+1}} \geq B_1^{\frac{1}{2}}A_k^{k-1}B_1^{\frac{1}{2}} \geq B_1^{\frac{1}{2}}A_{k-1}^{k-1}B_1^{\frac{1}{2}}, \tag{3.17}$$

since the first inequality is obtained by putting $\alpha_1 = k-1$ and $\alpha_2 = k$ in (3.16), and the second holds since (3.12) holds for $n = k-1$ by the inductive assumption. (3.17) yields the following (3.18):

$$(|T^*||T^k|^2|T^*|)^{\frac{k}{k+1}} \geq |T^*||T^{k-1}|^2|T^*|. \tag{3.18}$$

Let $T = U|T|$ be the polar decomposition of T, then $T^* = U^*|T^*|$ is the polar decomposition of T^*. Here we have

$$\begin{aligned}
|T^{k+1}|^{\frac{2k}{k+1}} &= (T^*|T^k|^2T)^{\frac{k}{k+1}} \\
&= (U^*|T^*||T^k|^2|T^*|U)^{\frac{k}{k+1}} \\
&= U^*(|T^*||T^k|^2|T^*|)^{\frac{k}{k+1}}U \\
&\geq U^*|T^*||T^{k-1}|^2|T^*|U \qquad \text{by (3.18)} \\
&= T^*|T^{k-1}|^2T \\
&= |T^k|^2,
\end{aligned}$$

so that it is proved that (3.12) holds for $n = k$. (3.13) can be proved in the same way as (3.12), so that we omit the proof.

Proof of (ii). The first inequality of (ii) has been already proved in (3.14), and the second can be proved in the same way as the first. □

Proof of Theorem 2. Put $A_n = |T^n|^{\frac{2}{n}}$ and $B_n = |T^{n*}|^{\frac{2}{n}}$ for each integer n, then T belongs to class $w\mathrm{A}(s,t)$ if and only if

$$(B_1^{\frac{t}{2}}A_1^sB_1^{\frac{t}{2}})^{\frac{t}{s+t}} = (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t} = B_1^t \tag{3.19}$$

and

$$A_1^s = |T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}} = (A_1^{\frac{s}{2}}B_1^tA_1^{\frac{s}{2}})^{\frac{s}{s+t}} \tag{3.20}$$

by the definition. Now T belongs to class wA since

$$\text{class } w\text{A} = \text{class } w\text{A}(1,1) \supseteq \text{class } w\text{A}(s,t)$$

for $s \in (0,1]$ and $t \in (0,1]$ by (ii) of Theorem C.1, so that by (ii) of Theorem 1,

$$A_n \geq A_1 \tag{3.21}$$

and

$$B_1 \geq B_n \tag{3.22}$$

hold for all positive integer n. Hence we have

$$A_n^s \geq (A_n^{\frac{s}{2}} B_1^t A_n^{\frac{s}{2}})^{\frac{s}{s+t}} \geq (A_n^{\frac{s}{2}} B_n^t A_n^{\frac{s}{2}})^{\frac{s}{s+t}}. \tag{3.23}$$

The first inequality in (3.23) is obtained by applying (ii) of Lemma 5 to (3.20) and (3.21) since A_1 and A_n satisfy the condition

$$(\bigstar) \qquad \text{if } \lim_{k\to\infty} A_1^{\frac{1}{2}} x_k = 0 \text{ and } \lim_{k\to\infty} A_n^{\frac{1}{2}} x_k \text{ exists, then } \lim_{k\to\infty} A_n^{\frac{1}{2}} x_k = 0,$$

and the second holds by (3.22) and Löwner-Heinz theorem. (3.23) yields the following (3.24):

$$|T^n|^{\frac{2s}{n}} \geq (|T^n|^{\frac{s}{n}} |T^{n*}|^{\frac{2t}{n}} |T^n|^{\frac{s}{n}})^{\frac{\frac{s}{n}}{\frac{s}{n}+\frac{t}{n}}}. \tag{3.24}$$

The following (3.25) can be obtained in the same way as (3.24):

$$(|T^{n*}|^{\frac{t}{n}} |T^n|^{\frac{2s}{n}} |T^{n*}|^{\frac{t}{n}})^{\frac{\frac{t}{n}}{\frac{s}{n}+\frac{t}{n}}} \geq |T^{n*}|^{\frac{2t}{n}}, \tag{3.25}$$

so that T^n belongs to class $w\text{A}(\frac{s}{n}, \frac{t}{n})$ by the definition. □

Proof of Corollary 3. If T belongs to class $w\text{A}(\frac{1}{2}, \frac{1}{2})$, then T^n belongs to class $w\text{A}(\frac{1}{2n}, \frac{1}{2n})$ by Theorem 2, so that T^n belongs to class $w\text{A}(\frac{1}{2}, \frac{1}{2})$ by (ii) of Theorem C.1. Hence the proof is complete since class $w\text{A}(\frac{1}{2}, \frac{1}{2})$ coincides with the class of w-hyponormal operators. □

4 Concluding remarks

Remark 1. $(B^{\frac{\beta_0}{2}}A^{\alpha_0}B^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}} \geq B^{\beta_0}$ and $A^{\alpha_0} \geq (A^{\frac{\alpha_0}{2}}B^{\beta_0}A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta_0}}$ in the assumptions of (i) and (ii) of Proposition 4 are mutually equivalent in case both A and B are invertible. In fact, by applying Lemma F to the right-hand side of the second inequality, we have

$$A^{\alpha_0} \geq (A^{\frac{\alpha_0}{2}}B^{\beta_0}A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta_0}} = A^{\frac{\alpha_0}{2}}B^{\frac{\beta_0}{2}}(B^{\frac{\beta_0}{2}}A^{\alpha_0}B^{\frac{\beta_0}{2}})^{\frac{-\beta_0}{\alpha_0+\beta_0}}B^{\frac{\beta_0}{2}}A^{\frac{\alpha_0}{2}},$$

so that the first inequality is obtained. But it is pointed out in [20] that they are not equivalent in general if either A or B are not invertible. In fact, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ satisfy the second inequality, but do not satisfy the first.

Remark 2. Lemma 5 can be proved easily in case A, B and C are invertible. In fact, (i) can be proved as follows: By Lemma F, $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ and $(C^{\frac{r}{2}}A^pC^{\frac{r}{2}})^{\frac{r}{p+r}} \geq C^r$ are equivalent to $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ $A^p \geq (A^{\frac{p}{2}}C^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$, respectively, so that the first inequality implies the second by the assumption $B \geq C$ and Löwner-Heinz theorem. (ii) can be proved similarly.

And one might expect that (ii) of Lemma 5 holds without the condition $(*)$. But there exists a counterexample. Put

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},$$

then $A \geq B$ and $N(A) \subsetneqq N(B)$, so that A and B do not satisfy the condition $(*)$. And for each $p > 0$ and $r \in (0,1]$,

$$B^r = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \geq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = (B^{\frac{r}{2}}C^pB^{\frac{r}{2}})^{\frac{r}{p+r}}$$

but

$$A^r = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \not\geq \begin{pmatrix} 0 & 0 \\ 0 & 2^{\frac{pr}{p+r}} \end{pmatrix} = (A^{\frac{r}{2}}C^pA^{\frac{r}{2}})^{\frac{r}{p+r}}.$$

References

[1] A.Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integral Equations Operator Theory **13** (1990), 307–315.

[2] A.Aluthge and D.Wang, *An operator inequality which implies paranormality*, Math. Inequal. Appl. **2** (1999), 113–119.

[3] A.Aluthge and D.Wang, *Powers of p-hyponormal operators*, J. Inequal. Appl. **3** (1999), 279–284.

[4] A.Aluthge and D.Wang, *w-Hyponormal operators*, Integral Equations Operator Theory **36** (2000), 1–10.

[5] A.Aluthge and D.Wang, *w-Hyponormal operators II*, Integral Equations Operator Theory **37** (2000), 324–331.

[6] T.Ando, *Operators with a norm condition*, Acta Sci. Math. (Szeged) **33** (1972), 169–178.

[7] M.Cho, *On spectra of* AB *and* BA, Proc. KOTAC **3** (2000), 15–19.

[8] M.Cho, T.Huruya and Y.O.Kim, *A note on w-hyponormal operators*, to appear in J. Inequal. Appl.

[9] R.G.Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413–415.

[10] M.Fujii, *Furuta's inequality and its mean theoretic approach*, J. Operator Theory **23** (1990), 67–72.

[11] M.Fujii, D.Jung, S.H.Lee, M.Y.Lee and R.Nakamoto, *Some classes of operators related to paranormal and log-hyponormal operators*, Math. Japon. **51** (2000), 395–402.

[12] T.Furuta, $A \geq B \geq 0$ *assures* $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ *for* $r \geq 0$, $p \geq 0$, $q \geq 1$ *with* $(1+2r)q \geq p+2r$, Proc. Amer. Math. Soc. **101** (1987), 85–88.

[13] T.Furuta, *An elementary proof of an order preserving inequality*, Proc. Japan Acad. Ser. A Math. Sci. **65** (1989), 126.

[14] T.Furuta, *Extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Algebra Appl. **219** (1995), 139–155.

[15] T.Furuta, M.Ito and T.Yamazaki, *A subclass of paranormal operators including class of log-hyponormal and several related classes*, Sci. Math. **1** (1998), 389–403.

[16] T.Furuta and M.Yanagida, *On powers of p-hyponormal operators*, Sci. Math. **2** (1999), 279–284.

[17] T.Furuta and M.Yanagida, *On powers of p-hyponormal and log-hyponormal operators*, J. Inequal. Appl. **5** (2000), 367–380.

[18] F.Hansen, *An operator inequality*, Math. Ann. **246** (1979/80), 249–250.

[19] T.Huruya, *A note on p-hyponormal operators*, Proc. Amer. Math. Soc. **125** (1997), 3617–3624.

[20] M.Ito, *Some classes of operators associated with generalized Aluthge transformation*, SUT J. Math. **35** (1999), 149–165.

[21] M.Ito, *Several properties on class A including p-hyponormal and log-hyponormal operators*, Math. Inequal. Appl. **2** (1999), 569–578.

[22] M.Ito, *Generalizations of the results on powers of p-hyponormal operators*, to appear in J. Inequal. Appl.

[23] E.Kamei, *A satellite to Furuta's inequality*, Math. Japon. **33** (1988), 883–886.

[24] Y.O.Kim, *An application of Furuta inequality*, Nihonkai Math. J. **10** (1999), 195–198.

[25] K.Tanahashi, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc. **124** (1996), 141–146.

[26] K.Tanahashi, *On log-hyponormal operators*, Integral Equations Operator Theory **34** (1999), 364–372.

[27] T.Yamazaki, *Extensions of the results on p-hyponormal and log-hyponormal operators by Aluthge and Wang*, SUT J. Math. **35** (1999), 139–148.

[28] T.Yamazaki, *On powers of class* $A(k)$ *operators including p-hyponormal and log-hyponormal operators*, Math. Inequal. Appl. **3** (2000), 97–104.

[29] T.Yoshino, *The p-hyponormality of the Aluthge transform*, Interdiscip. Inform. Sci. **3** (1997), 91–93.